Cambridge Integral Bee Sample Solutions

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Introduction

Similarly to the hints, each page will be for a question to avoid accidentally spoiling it for yourself. The aim of the sample problems are to introduce/teach the main techniques which will be used in the competitions. Competitions likely will have more definite integrals than indefinite integrals as I feel like definite integrals can give a lot more variety in the techniques involved to evaluate them. DUTIS will be used as an abbreviation for differentiation under the integral sign:

$$\frac{d}{dt}\left(\int_a^b f(x,t)\mathrm{d}x\right) = \int_a^b \frac{\partial}{\partial t}\left(f(x,t)\right)\mathrm{d}x$$

For those who did the competition in 2020, the sample problems are the same as the ones then. Some solutions have been updated to have some extra comments or alternative solutions too.

Question 1

Let $I(t) = \displaystyle\int_0^1 \frac{x^t - 1}{\ln x}\mathrm{d}x$.

Using DUTIS,

$$I'(t) = \int_0^1 x^t \mathrm{d}x = \left[\frac{x^{t+1}}{t+1}\right]_0^1 = \frac{1}{t+1}$$

Now that means that $I(t) = \ln(t+1) + c$. Note that $I(0) = 0 = \ln 1 + c = c$ therefore $I(t) = \ln(t+1)$ so $I(5) = \ln 6$ in particular.

Due to the parameter required in DUTIS, you get a generalisation of the integral which is neat.
For the alternative solution, we have upon writing it as a double integral:

$$I = \int_0^1 \int_0^5 x^y \mathrm{d}y\mathrm{d}x$$

Now that it's a double integral, we can swap the order of integration. Doing this gives us

$$I = \int_0^5 \int_0^1 x^y \mathrm{d}x\mathrm{d}y = \int_0^5 \frac{\mathrm{d}y}{y+1} = \ln 6$$

This method can also be applied to the more general integral $I(t)$ too. Usually this double integral trick works in a similar way to DUTIS (friendly with parameters) and will generally be quite useful whenever there's two functions of the same type being subtracted from each other e.g $f(tx) - f(sy)$.

Question 2

This is the well known Dirichlet integral. The solution which is presented here is using DUTIS. For those who know about Laplace transforms, it can be more easily evaluated with that.

Let $I(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} \mathrm{d}x$. The motivation for this is what was in the hints; differentiating it gets rid of the x in denominator and the integral is convergent upon differentiating. Another key bonus is that as $t \to \infty$, the integral approaches 0 which gives a way to evaluate the constant of integration.

Using DUTIS, $I'(t) = -\int_0^\infty e^{-tx} \sin x \mathrm{d}x$ which is a standard integral - it can be evaluated by integrating by parts twice. For those who are aware of Laplace transforms, you can just look up the Laplace transform of $\sin t$ for this and it gives $I'(t) = -\frac{1}{t^2+1}$.

Integrating this, $I(t) = c - \arctan t$. Noting that $\lim_{t \to \infty} I(t) = 0$ from above, $c - \frac{\pi}{2} = 0$ so $I(t) = \frac{\pi}{2} - \arctan t$. The required integral is then $I(0) = \frac{\pi}{2}$.

Question 3

Let $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \mathrm{d}x$.

The main idea in this is the reflection property,

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(a+b-x)\mathrm{d}x$$

Use the substitution $u = \frac{\pi}{2} - x$ so that $\mathrm{d}u = -\mathrm{d}x$ and the limits change from $\frac{\pi}{2}$ to 0.

Now noting that

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$$

and $\sin\left(\frac{\pi}{2} - x\right) = \cos x$ and $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \mathrm{d}x$.

Adding this to the original form of I,

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \mathrm{d}x = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \frac{\pi}{2}$$

Therefore $I = \frac{\pi}{4}$

Question 4

Let $I = \int_0^\infty \frac{\ln x}{x^2+1} \mathrm{d}x$. Let $u = \frac{1}{x}$.
Then, the limits become from ∞ to 0 and $\frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{x^2}$ so that $\mathrm{d}x = -\frac{\mathrm{d}u}{u^2}$.

Putting all of this information together,

$$I = \int_\infty^0 \frac{\ln\left(\frac{1}{u}\right)}{1+\frac{1}{u^2}} - \frac{\mathrm{d}u}{u^2} = \int_0^\infty \frac{-\ln u}{1+u^2} \mathrm{d}u = -I$$

Since $I = -I, I = 0$.

Another method for this integral is to let $x = \tan\theta$ and noting that $\ln(\tan\theta) = \ln(\sin\theta) - \ln(\cos\theta)$, can you see why?

Question 5

Let $I = \int_1^2 \frac{\ln x}{2 - 2x + x^2} \mathrm{d}x = \int_1^2 \frac{\ln x}{(x-1)^2 + 1} \mathrm{d}x.$

Now make the substitution $u = x - 1$ so that the limits become from 0 to 1 and $\mathrm{d}u = \mathrm{d}x$. Then the integral becomes

$$I = \int_0^1 \frac{\ln(u+1)}{u^2+1} \mathrm{d}u$$

This integral is known as Serret's integral. With the $x^2 + 1$ denominator, it looks ripe for a tan substitution; let $u = \tan\theta$ so that $\mathrm{d}u = \sec^2\theta \mathrm{d}\theta$ and the limits become from 0 to $\frac{\pi}{4}$. Then,

$$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan\theta) \mathrm{d}\theta$$

Now the reflection property from earlier can be applied; let $t = \frac{\pi}{4} - \theta$. In particular,

$$\tan\left(\frac{\pi}{4} - t\right) = \frac{\tan\left(\frac{\pi}{4}\right) - \tan t}{1 + \tan\left(\frac{\pi}{4}\right)\tan t} = \frac{1 - \tan t}{1 + \tan t}$$

Given that,

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan t}{1 + \tan t}\right) \mathrm{d}t \\ &= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan t}\right) \mathrm{d}t \\ &= \int_0^{\frac{\pi}{4}} \ln 2 \mathrm{d}t - \int_0^{\frac{\pi}{4}} \ln(1 + \tan t) \mathrm{d}t \\ &= \frac{\pi \ln 2}{4} - I \end{aligned}$$

So now, $I = \frac{\pi \ln 2}{8}$.
If you'd like more practice with DUTIS, you can try Serret's integral using it.

Question 6

This integral is best done by considering the power series expansion of $\ln(1-x)$.

$$\begin{aligned} I = \int_0^1 \frac{\ln(1-x)}{x} \mathrm{d}x &= \int_0^1 -\sum_{n=1}^{\infty} \frac{x^{n-1}}{n} \mathrm{d}x \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \mathrm{d}x \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \left[\frac{x^n}{n} \right]_0^1 \\ &= -\sum_{n=1}^{\infty} \frac{1}{n^2} = -\zeta(2) \end{aligned}$$

by using the definition of the Riemann zeta function. Some of you may be aware that this value is $\frac{\pi^2}{6}$ so $I = -\frac{\pi^2}{6}$.

Question 7

This is the gamma function. The approach here is to integrate by parts, let $u = x^{n-1}$, $\frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = e^{-x}$ so $v = -e^{-x}$

Let $I_n = \int_0^\infty x^{n-1}e^{-x}\mathrm{d}x$. Then,

$$\begin{aligned} I_n &= \left[-e^{-x}x^n\right]_0^\infty + \int_0^\infty (n-1)x^{n-2}e^{-x}\mathrm{d}x \\ &= (n-1)I_{n-1} \end{aligned}$$

Noting that $I_n = 1$, this recurrence relation means that $I_n = (n-1)!$. Of course, as this integral is defined for non integer arguments, it serves as an extension of the factorial function from integer to real and complex arguments. $I_n = \Gamma(n)$ and the working out above holds for all $n > 0$ so $\Gamma(n) = (n-1)\Gamma(n-1)$. There are many different representations of the gamma function such as the Weierstrass and Euler product forms and these two properties, along with the fact that $\ln\Gamma(x)$ is convex, uniquely determine the Gamma function due to the Bohr-Mollerup theorem.

Question 8

In this question we'll combine the method with of Question 6 and the function in Question 7. Right now, the power series resulting from expanding a constituent part of the integrand are all divergent so first, divide the top and bottom by e^x.

$$
\begin{aligned}
\int_0^\infty \frac{x^{s-1}}{e^x - 1}\mathrm{d}x &= \int_0^\infty \frac{x^{s-1}e^{-x}}{1 - e^{-x}}\mathrm{d}x \\
&= \int_0^\infty x^{s-1} \sum_{k=1}^{\infty} e^{-kx}\mathrm{d}x \\
&= \sum_{k=1}^{\infty} \int_0^\infty x^{s-1}e^{-kx}\mathrm{d}x \\
&= \sum_{k=1}^{\infty} \frac{1}{k^s} \int_0^\infty u^{s-1}e^{-u}\mathrm{d}u \quad (u = kx) \\
&= \left(\int_0^\infty x^{s-1}e^{-x}\mathrm{d}x\right) \sum_{k=1}^{\infty} \frac{1}{k^s} = \Gamma(s)\zeta(s)
\end{aligned}
$$

where in the last step, $\Gamma(s)$ was taken out of the sum as a constant independent of k; it only depends on s.

Question 9

This integral is done by handling the floor function so that it can be put into a form that can be integrated. Note that if $k \leq x < k+1$ then $\lfloor x \rfloor = k$. With this in mind, write the integral in this form:

$$\begin{aligned}
\int_1^\infty \frac{x - \lfloor x \rfloor}{x^4} \mathrm{d}x &= \int_0^\infty \frac{\mathrm{d}x}{x^3} - \int_1^\infty \frac{\lfloor x \rfloor}{x^4} \mathrm{d}x \\
&= \left[-\frac{1}{2x^2} \right]_1^\infty - \sum_{k=1}^\infty \int_k^{k+1} \frac{k}{x^4} \mathrm{d}x \\
&= \frac{1}{2} - \sum_{k=1}^\infty \left[-\frac{k}{3x^3} \right]_k^{k+1} \\
&= \frac{1}{2} - \sum_{k=1}^\infty \left[\frac{k}{3k^3} - \frac{k}{3(k+1)^3} \right] \text{(telescoping sum)} \\
&= \frac{1}{2} - \left(\frac{1}{3 \cdot 1^3} - \frac{1}{3 \cdot 2^3} + \frac{2}{3 \cdot 2^3} - \frac{2}{3^3} + \frac{3}{3 \cdot 3^3} + \dots \right) \\
&= \frac{1}{2} - \frac{1}{3} \left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots \right) \\
&= \frac{1}{2} - \frac{1}{3} \zeta(3)
\end{aligned}$$

Question 10

This problem is actually from a meme here, students at Nanjing University of Aeronautics and Astronautics had to solve this integral for access to the wifi. So now, if you ever find yourself there, after reading this you'll be able to get onto their wifi!

Anyway, I think this is best done by considering the two parts of the integrand. The function in first part, $f(x) = x^3 \cos\left(\frac{x}{2}\right)\sqrt{4-x^2}$, is an odd function as $f(-x) = f(x)$ so that

$$\int_{-2}^{2} x^3 \cos\left(\frac{x}{2}\right)\sqrt{4-x^2}\mathrm{d}x = 0$$

For the second part, the integrand is even so it isn't obliterated by integrating from -2 to 2. To evaluate it, a sine or cosine substitution can be made; an alternative way is to note that if $y = \sqrt{4-x^2}$, $x^2 + y^2 = 4$ so that this is half the area of a semicircle of radius 2. Therefore,

$$\int_{-2}^{2} \frac{1}{2}\sqrt{4-x^2}\mathrm{d}x = \pi$$

which is the value of the whole integral. The wifi password then is 3.141592653.

Question 11

Another method for this integral is to let $u = \tan x$ which is more direct but this solution will go over the method which is outlined in the hints.

Upon multiplying top and bottom by $\cos x$, the desired integral is $I = \int \frac{\cos x}{\sin x + \cos x} \mathrm{d}x$ and let $J = \int \frac{\sin x}{\sin x + \cos x} \mathrm{d}x$.

Now

$$\begin{aligned} I + J &= \int \frac{\sin x + \cos x}{\sin x + \cos x} \mathrm{d}x = \int \mathrm{d}x = x + c \\ I - J &= \int \frac{\cos x - \sin x}{\sin x + \cos x} \mathrm{d}x = \ln(\sin x + \cos x) + c \end{aligned}$$

so that $I = \frac{1}{2}(x + \ln(\sin x + \cos x)) + c$.

Question 12

There isn't much in this one other that once you realise to differentiate $\sqrt{\tan x}$, you get

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{\tan x}) = \frac{\sec^2(x)}{2\sqrt{\tan x}} = \frac{\sqrt{\tan x}}{2\sin x\cos x} = \frac{\sqrt{\tan x}}{\sin(2x)}$$

which is exactly the integrand.

A different method suggested by Sharky Kesa#9845 on discord is to make a substitution which makes the numerator disappear - that might help you work this one out more easily.

Question 13

This integral is here to remind you to consider modifying the integrand to make the integral easier. For this one, just divide the top and bottom by e^x and then the derivative of the top is equal to the derivative of the bottom:

$$\int \frac{\mathrm{d}x}{1+e^x} = \int \frac{e^{-x}}{1+e^{-x}}\mathrm{d}x = -\ln(1+e^{-x}) + c$$

Question 14

First, let $u = \ln x$ so that $\mathrm{d}u = \dfrac{\mathrm{d}x}{x}$, then the integral becomes

$$\begin{aligned}
\int \frac{\mathrm{d}x}{1+\sin^2 x} &= \int \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\mathrm{d}x \\
&= \int \frac{\sec^2 x}{1 + 2\tan^2 x}\mathrm{d}x \\
&= \int \frac{\mathrm{d}u}{1+2u^2}\mathrm{d}u \quad (u = \sqrt{2}\tan x) \\
&= \frac{1}{\sqrt{2}}\arctan(\sqrt{2}u) + c = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan(\ln x)) + c
\end{aligned}$$

Question 15

This relies on the sum of cubes to provide a partial fraction representation; $x^3+1=(x+1)(x^2-x+1)$. The general representation for this fraction is

$$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

To find the constants, multiply through by the two factors so that

$$1=A(x^2-x+1)+(Bx+C)(x+1)$$

Let $x=-1$ so that $A=\frac{1}{3}$. Then, considering the x^2 coefficient, $B=-A=-\frac{1}{3}$ and considering the constant term, $A+C=1$ so that $C=\frac{2}{3}$. Then

$$\begin{aligned}
\int\frac{\mathrm{d}x}{x^3+1} &= \int\frac{1}{3(x+1)}+\frac{2-x}{3(x^2-x+1)}\mathrm{d}x \\
&= \frac{1}{3}\ln(x+1)-\frac{1}{6}\int\frac{2x-1}{x^2-x+1}\mathrm{d}x+\frac{1}{2}\int\frac{\mathrm{d}x}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{1}{3}\ln(x+1)-\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2}\int\frac{\frac{\sqrt{3}}{2}\sec^2 u}{\frac{3}{4}\sec^2 u}\mathrm{d}u \quad \left(\frac{\sqrt{3}}{2}\tan u=x-\frac{1}{2}\right) \\
&= \frac{1}{3}\ln(x+1)-\frac{1}{6}\left(\ln(x^2-x+1)+2\sqrt{3}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right)+c
\end{aligned}$$

Quite messy!

Question 16

The idea here is to use the Weierstrass substitution, $t = \tan\left(\frac{x}{2}\right)$ so $\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$ and $\mathrm{d}x = \frac{2\mathrm{d}t}{1+t^2}$ - have a go at deriving these if you're unfamiliar with it. Now, upon making this substitution,

$$\begin{aligned}
\int \frac{\mathrm{d}x}{2+\sin x} &= \int \frac{2\mathrm{d}t}{2t+2t^2+2} \\
&= \int \frac{\mathrm{d}t}{(t+\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{2}{\sqrt{3}} \arctan\left(\frac{2t+1}{\sqrt{3}}\right) + c \quad \text{(after a substitution like the one in Q15)} \\
&= \frac{2}{\sqrt{3}} \arctan\left(\frac{2\tan\left(\frac{x}{2}\right)+1}{\sqrt{3}}\right) + c
\end{aligned}$$

Question 17

This is just a fun little one to make sure you're aware of converting things to base e. First, note that $x = e^{\ln x}$ so that $x^a = e^{a \ln x}$ for any real number a.

Now $x^{\frac{1}{\ln x}} = e$ so the integral is just $ex + c$.

Question 18

You can go for this with Weierstrass substitution if you wanted but sometimes with trig integrals it's better to mess around with it for a bit. A nice idea to keep in mind is if you can exploit trig identities like $1 - \sin^2 x = \cos^2 x$ to simplify that denominator into one function so difference of two squares is good to keep in mind. This question is a perfect example of that:

$$\begin{aligned}
\int \frac{\mathrm{d}x}{1 - \sin x} &= \int \frac{1 + \sin x}{1 - \sin^2 x} \mathrm{d}x \\
&= \int \frac{1 + \sin x}{\cos^2 x} \mathrm{d}x \\
&= \int \sec^2 x \mathrm{d}x + \int \sec x \tan x \mathrm{d}x = \sec x + \tan x + c
\end{aligned}$$

Question 19

First, I'd like to turn down the x^2 terms with a $u = x^2$ substitution. $\mathrm{d}u = 2x\mathrm{d}x$ so the integral becomes $\frac{1}{2}\int\sqrt{\frac{1-u}{1+u}}\mathrm{d}u$. Now let $u = \cos(2v)$ so that $\mathrm{d}u = -2\sin(2v)\mathrm{d}v$ and more importantly, $1-u = 2\sin^2 v$; $1+u = 2\cos^2 v$. Putting all of this into the integral gives

$$\begin{aligned}
\frac{1}{2}\int\sqrt{\frac{1-u}{1+u}}\mathrm{d}u &= -\int\frac{\sin v}{\cos v}\sin(2v)\mathrm{d}v \\
&= -2\int\sin^2 v\mathrm{d}v \\
&= \int\cos(2v)-1\mathrm{d}v \\
&= \frac{1}{2}(\sin(2v)-2v)+c \\
&= \frac{1}{2}(\sin(\cos^{-1}(x^2))-\cos^{-1}(x^2))+c
\end{aligned}$$

As there's some sines and inverse cosines mixed here, that's a big algebraic expression so if you did it another way, you'd get an equivalent answer which looks quite different, for example on Symbolab's integral calculator. You can confirm your answer by running it through a derivative calculator.

Question 20

For this integral, I turned it into a rational function which can be dealt with using a tan sub. This might not be the ideal way and, like question 19, you might get an answer which looks a lot different to answers you see online. First, take an e^x out of the top:

$$\begin{aligned}
I &= \int \frac{e^{2x}+e^{3x}}{e^{2x}+1} e^x \mathrm{d}x \\
&= \int \frac{u^3+u^2}{u^2+1} \mathrm{d}u \quad (u = e^x) \\
&= \int \frac{u^3}{u^2+1} \mathrm{d}u + \int \mathrm{d}u - \int \frac{\mathrm{d}u}{u^2+1} \\
&= u - \arctan u + \int \tan^3 v \mathrm{d}v \quad (u = \tan v) \\
&= e^x - \arctan e^x + \int \tan v(\sec^2 v - 1)\mathrm{d}v \\
&= e^x - \arctan e^x + \int \tan v \sec^2 v \mathrm{d}v - \int \tan v \mathrm{d}v \\
&= e^x - \arctan e^x + \frac{1}{2}\sec^2 v \tan v + \ln|\cos v| + c \\
&= e^x - \arctan e^x + \frac{1}{2}(\sec^2(\arctan(e^x))) + \ln|\cos(\arctan(e^x))| + c
\end{aligned}$$

Like the last question, you might get some different answers for this but check derivative calculator to make sure you're right.

An alternative solution suggested by George!#1242, you can do the 'adding zero' trick where you write u^3 integral as $u^3 + u - u$ which greatly simplifies the calculation.